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SERIES OF INDEPENDENT RANDOM VARIABLES IN REARRANGEMENT INVARIANT SPACES: AN OPERATOR APPROACH

BY

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ABSTRACT

This paper studies series of independent random variables in rearrangement invariant spaces X on $[0,1]$. Principal results of the paper concern such series in Orlicz spaces $\exp(L_p)$, $1 \leq p \leq \infty$ and Lorentz spaces Λ_ψ. One by-product of our methods is a new (and simpler) proof of a result due to W. B. Johnson and G. Schechtman that the assumption $L_p \subset X$, $p < \infty$ is sufficient to guarantee that convergence of such series in X (under the side condition that the sum of the measures of the supports of all individual terms does not exceed 1) is equivalent to convergence in X of the series of disjoint copies of individual terms. Furthermore, we prove the converse (in a certain sense) to that result.

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1. Introduction

It follows from the classical Khintchine Inequality that for all $p \in [1,\infty)$, the Rademacher system $\{r_n\}_{n=1}^\infty$, given by $r_n(t) = \operatorname{sgn}\sin(2^n\pi t)$, $t \in [0,1)$ in the L_p-spaces on the interval $[0,1]$ (equipped with Lebesgue measure λ) is equivalent to the unit vector basis $\{e_n\}_{n=1}^\infty$ of l_2, or equivalently to the sequence of disjoint translates $\bar{r}_n(t) := r_n(t-n+1)$ in $L_2(0,\infty)$. Similarly, it follows from a remarkable inequality due to H. P. Rosenthal [R] for sequences $\{f_n\}_{n=1}^\infty$ of independent mean zero random variables in $L_p[0,1]$, $p \geq 2$ that the mapping $f_k \to \bar{f}_k$, where $\bar{f}_k(t) := f_k(t-k+1)\chi_{[k-1,k)}(t)$, $t \in \mathbb{R}$ extends to an isomorphism between the closed linear span $[f_k]_{k=1}^\infty$ (taken in $L_p[0,1]$) and the closed linear span $[\bar{f}_k]_{k=1}^\infty$ (taken in $L_p[0,\infty) \cap L_2[0,\infty)$). An extension of Rosenthal's Inequality to Lorentz spaces $L_{p,q}$, $2 < p < \infty$, $0 < q < \infty$ is given in [CD]. A further significant generalization to the class of rearrangement invariant (=r.i.) spaces is due to W. B. Johnson and G. Schechtman [JS] who introduced r.i. spaces Y_X and Z_X on $[0,\infty)$ linked with a given r.i. space X on $[0,1]$ and showed that any finite sequence $\{f_k\}_{k=1}^n$ of independent mean zero (respectively, positive) random variables in X is equivalent (uniformly in n) to the sequence of its disjoint translates in Y_X (respectively, Z_X), provided that X contains an L_p-space for some $p < \infty$. A key tool in their proof of this equivalence is the well-known tail probability inequality due to Hoffmann–Jørgensen [H-J] (for interesting strengthening of this inequality we refer to the recent papers [HM] and [KN]). In particular, it immediately follows from results of [JS] that if the sequence $\{f_k\}_{k=1}^\infty$ of independent random variables satisfies, in addition, the assumption that for all $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \lambda(\{f_k \neq 0\}) \leq 1, \tag{1.1}$$

then the correspondence $f_k \leftrightarrow \bar{f}_k$, $k \geq 1$ between the sequence $\{f_k\}_{k=1}^\infty$ and the disjointly supported sequence $\{\bar{f}_k\}_{k=1}^\infty$ of equimeasurable copies of $\{f_n\}_{n=1}^\infty$ extends to an isomorphism between the closed linear spans $[f_k]_{k=1}^\infty$ and $[\bar{f}_k]_{k=1}^\infty$ in X, provided that X contains $L_p[0,1]$ for some $p < \infty$.

The main question studied in the present paper is the following: *for which r.i. spaces X and Y on $[0,1]$ does there exist a constant $C = C(X,Y) > 0$ such that for every sequence $\{f_k\}_{k=1}^n \subset X$ of independent random variables satisfying (1.1), it follows that*

$$\Big\| \sum_{k=1}^{n} f_k \Big\|_Y \leq C \Big\| \sum_{k=1}^{n} \bar{f}_k \Big\|_X ? \tag{1.2}$$

We recall that a complete characterization of rearrangement invariant spaces E on $[0,1]$ for which the Khintchine Inequality holds is due to V. A. Rodin and E. M. Semenov [RS] (see also [LT], pp. 134–136). Consider the family of Orlicz spaces $\exp(L_p) = L_{N_p}$, $N_p(t) := \exp|t|^p - 1$, $t \in \mathbb{R}$, $p \geq 1$. It is shown in [RS] that the correspondence $r_n \leftrightarrow e_n$, $n \geq 1$ extends to an isomorphism between the closed linear span $[r_n]_{n=1}^\infty$ in E and $[e_n]_{n=1}^\infty$ in l_2 if and only if E contains the separable part of the space $\exp(L_2)$.

It is proved here that in the setting that $X = Y$, unlike the situation with the Khintchine Inequality, there is no "minimal" r.i. space E such that for every r.i. space $X \supset E$ the correspondence $f_k \leftrightarrow \bar{f}_k$, $k \geq 1$ for all sequences $\{f_k\}_{k=1}^\infty$ of independent random variables satisfying (1.1) extends to an isomorphism between the closed linear spans $[f_k]_{k=1}^\infty$ and $[\bar{f}_k]_{k=1}^\infty$ in X. In fact, we show that if E is such an r.i. space, then it contains an L_p-space for some $p < \infty$. This result is the converse (in a certain sense) to [JS], Theorem 1.

We study the above question for the settings when $X = \exp(L_p)$, $1 \leq p \leq \infty$ and when X is an arbitrary Lorentz space $\Lambda_\psi[0,1]$. In the first case, we show that for a fixed $p \in [1, \infty]$ and $X = \exp(L_p)$, the set of all r.i. spaces Y for which the inequality (1.2) holds has a unique minimal element, which is the Orlicz space L_{M_q}, where $M_q(t) := e^{|t| \ln^{1/q}(e+|t|)} - 1$, $t \in \mathbb{R}$, and $q = p/(p-1)$. In the second setting, we give a complete characterization of those concave functions ψ such that the correspondence $f_k \leftrightarrow \bar{f}_k$, $k \geq 1$ for all sequences $\{f_k\}_{k=1}^\infty$ of independent random variables satisfying (1.1) extends to an isomorphism between the closed linear spans $[f_k]_{k=1}^\infty$ and $[\bar{f}_k]_{k=1}^\infty$ in Lorentz space $\Lambda_\psi[0,1]$.

Our approach is based on the study of a certain linear operator $\mathcal{K}$ on $L_1[0,1]$ and is related to the approach previously developed by M. Sh. Braverman [Br] in his study of the Rosenthal Inequality in r.i. spaces, which, in turn, was inspired by earlier ideas and probabilistic constructions of V. M. Kruglov [K]. We spell out these connections in Section 3 below, after introducing all necessary definitions in Section 2. We study the main question in the Orlicz spaces $\exp(L_p)$ and the Lorentz spaces Λ_ψ in Sections 4 and 5 respectively. As a by-product of the work carried out in Section 5 for Lorentz spaces, we also present there the converse of the main result from [JS]. The latter application is partly based on the results of S. Montgomery-Smith and E. M. Semenov [MS] concerning random rearrangements, although our exposition is fairly self-contained. In Section 6, we show an easy way to discard the side condition (1.1) and extend our results to an arbitrary sequence of independent random variables. In particular, in the setting that $X = Y$, we recover (and complement) an earlier result from [JS] for the space Z_X. Our approach here consists in reducing the study of arbitrary sequences of independent random variables to those satisfying condition (1.1). We demonstrate the utility of our approach by strengthening recent results of S. Montgomery-Smith [M] concerning spaces Z_X^p, $p \in (1, \infty]$ (which generalize results of [JS] for the space Z_X). Finally, in Section 7, we present a necessary condition for an affirmative answer to the main question in the case when $X = Y$.

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Some of the results of this paper have been announced in [AS].

2. Definitions and preliminaries

We denote by $S(\Omega)(= S(\Omega, \mathcal{P}))$ the linear space of all measurable finite a.e. functions on a given measure space $(\Omega, \mathcal{P})$ equipped with the topology of convergence locally in measure.

A Banach space $(E, \|\cdot\|_E)$ of real-valued Lebesgue measurable functions on the interval $[0, \alpha)$, $0 < \alpha \leq \infty$ (with identification λ-a.e.) will be called **rearrangement invariant** if

(i) E is an ideal lattice, that is, if $y \in E$, and if x is any measurable function on $[0, \alpha)$ with $0 \leq |x| \leq |y|$ then $x \in E$ and $\|x\|_E \leq \|y\|_E$;

(ii) E is rearrangement invariant in the sense that if $y \in E$, and if x is any measurable function on $[0, \alpha)$ with $x^* = y^*$, then $x \in E$ and $\|x\|_E = \|y\|_E$.

Here, λ denotes Lebesgue measure and x^* denotes the non-increasing, right-continuous rearrangement of x given by

$$x^*(t) = \inf\{s \geq 0 \colon \lambda(\{| x |> s\}) \leq t\}, \quad t > 0.$$

For basic properties of rearrangement invariant spaces, we refer to the monographs [BS], [KPS], [LT].

The Köthe dual $E^\times$ of a rearrangement invariant space E on the interval $[0, \alpha)$ consists of all measurable functions y for which

$$\|y\|_{E^\times} := \sup\left\{ \int_0^\alpha |x(t)y(t)|dt \colon x \in E, \|x\|_E \leq 1 \right\} < \infty.$$

Basic properties of Köthe duality may be found in [KPS], [BS] (where the Köthe dual is called the **associate** space). If E^* denotes the Banach dual of E, it is known that $E^\times \subset E^*$ and $E^\times = E^*$ if and only if the norm $\|\cdot\|_E$ is order-continuous, i.e. from $\{x_n\} \subseteq E, x_n \downarrow_n 0$, it follows that $\|x_n\|_E \to 0$. We note that the norm $\|\cdot\|_E$ of the rearrangement invariant space E on $[0,\alpha)$ is order-continuous if and only if E is separable. The natural embedding of E into its Köthe bidual $E^{\times\times}$ is a surjective isometry if and only if E has the **Fatou property**, i.e. if it follows from $\{f_n\}_{n\geq 1} \subseteq E$, $f \in S[0,\alpha)$, $f_n \to f$ a.e. on $[0,\alpha)$ and $\sup_n \|f_n\|_E < \infty$, that $f \in E$ and $\|f\|_E \leq \liminf_{n\to\infty} \|f_n\|_E$. Such spaces are also called **maximal**. Somewhat weaker than the notion of Fatou property of an r.i. space E is the notion of a Fatou norm. If E is a r.i. Banach function space on $[0,\alpha)$, $0 < \alpha \leq \infty$, then the norm $\|\cdot\|_E$ on E is said to be a **Fatou norm**, if the unit ball of E is closed in E with respect to almost everywhere convergence. The norm on the r.i. space E is a Fatou norm if and only if the natural embedding of E into its Köthe bidual is an isometry.

For any r.i. space E on $[0,\alpha)$, the inclusions

$$L_1[0,\alpha) \cap L_\infty[0,\alpha) \subseteq E \subseteq L_1[0,\alpha) + L_\infty[0,\alpha)$$

hold with continuous embeddings. We denote the closure of $L_1[0,\alpha) \cap L_\infty[0,\alpha)$ in E by E^0. If $E \neq L_\infty$, then E^0 is a separable r.i. space.

If $A \subseteq [0,\infty)$ is (Lebesgue) measurable, we denote the indicator function of A by χ_A. Given the rearrangement invariant space E, the function

$$\phi_E(t) := \|\chi_A(\cdot)\|_E,$$

where the measurable set A satisfies $\lambda(A) = t$, is called the **fundamental function** of E.

3. The Kruglov property and the operator $\mathcal{K}$

Let f be a measurable function (random variable) on $[0,1]$. By $\pi(f)$ we will denote a random variable $\sum_{i=1}^N f_i$ where f_i are independent copies of f, and N is a Poisson random variable with parameter 1 independent of the sequence $\{f_i\}$. Another (equivalent) definition of $\pi(f)$ may be given via its characteristic function as follows,

$$\phi_{\pi(f)}(t) = \exp\left(\int_{-\infty}^{\infty} (e^{itx} - 1)d\mathcal{F}_f(x)\right),$$

where $\mathcal{F}_f$ is the distribution function of f [Br].

Everywhere in this section X stands for an r.i. space on $[0,1]$. It will be convenient to adopt the following terminology.

Definition 3.1: An r.i. space X is said to have the Kruglov property ($X \in \mathbb{K}$), if and only if

$$f \in X \Longleftrightarrow \pi(f) \in X.$$

This property has been studied and extensively used by M. Sh. Braverman [Br], who noted in particular that only the implication $f \in X \Longrightarrow \pi(f) \in X$ is non-trivial, since the implication $\pi(f) \in X \Longrightarrow f \in X$ is always satisfied (see [Br], p. 11).

We shall now define an operator $\mathcal{K}$ on $S([0,1],\lambda)$ which is closely linked with the Kruglov property. From a technical viewpoint, it is more convenient to assume that this operator takes its values in $S(\Omega,\mathcal{P})$, where $(\Omega,\mathcal{P}) := \prod_{k=0}^{\infty}([0,1],\lambda_k)$ (here, λ_k is Lebesgue measure on $[0,1]$ for every $k \geq 0$). Let $\{E_n\}$ be a sequence of pairwise disjoint subsets of $[0,1]$, $m(E_n) = 1/(e\cdot n!)$, $n \in \mathbb{N}$. For a given $f \in S([0,1],\lambda)$, we set

$$\mathcal{K}f(\omega_0,\omega_1,\omega_2,\ldots) := \sum_{n=1}^{\infty}\sum_{k=1}^{n} f(\omega_k)\chi_{E_n}(\omega_0). \tag{3.1}$$

Let also δ: $(\Omega,\mathcal{P}) \to ([0,1],\lambda)$ be a measure preserving isomorphism. For every $g \in S(\Omega,\mathcal{P})$, we set $T(g)(x) := g(\delta^{-1}x)$, $x \in [0,1]$. Note that T is a rearrangement-preserving mapping between $S(\Omega,\mathcal{P})$ and $S([0,1],\lambda)$. We shall be mainly interested in the operator $T\mathcal{K}$ acting on $S([0,1],\lambda)$ and by an abuse of language frequently refer to the latter operator as $\mathcal{K}$. From a certain viewpoint, our main object of study in this paper is the distribution of $\mathcal{K}f$ (for various classes of measurable functions $f \in S[0,1]$). Therefore, it will be convenient to adopt the following notation. If $f \in S([0,1],\lambda)$ and $\{f_{n,k}\}_{k=1}^{n}$ is a sequence of measurable functions on $[0,1]$ such that:

(i) the sequence $f_{n,1}, f_{n,2},\ldots,f_{n,n}, \chi_{E_n}$ is a sequence of independent random variables $\forall n \in \mathbb{N}$;

(ii) $\mathcal{F}_{f_{n,k}} = \mathcal{F}_f$, $\quad \forall n \in \mathbb{N},\ k = 1,2,\ldots,n$,

then we write

$$\mathcal{K}'f(x) := \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{n,k}(x)\chi_{E_n}(x), \quad x \in [0,1]. \tag{3.1$'$}$$

It is clear that the distribution function of $\mathcal{K}f$ $(= T\mathcal{K}f)$ is the same as the distribution function of $\mathcal{K}'f$. Frequently, again by an abuse of language, we shall also refer to $\mathcal{K}'f$ as $\mathcal{K}f$.

The main objective of this paper is to study the action of the positive linear operator $\mathcal{K}$ on various classes of r.i. spaces X.

Our first remark concerning the operator $\mathcal{K}$ immediately follows from the closed graph theorem (see also [Br], Lemma 1, pp. 11–12).

LEMMA 3.2: *If X and Y are r.i. spaces on $[0,1]$ and $\mathcal{K}f \in Y$ for every $f \in X$, then there exists $C > 0$ such that*

$$\|\mathcal{K}f\|_Y \leq C\|f\|_X.$$

Let $f \in S[0,1]$ and let $\mathcal{K}'f$ by defined by (3.1)$'$. The distribution function of the random variable $\mathcal{K}'f$ is given by

$$\mathcal{F}_{\mathcal{K}'f}(x) = \frac{1}{e}\left(\chi_{(0,\infty)}(x) + \mathcal{F}_f(x) + \sum_{l=2}^{\infty}\frac{1}{l!}(\mathcal{F}_f(x))^{*l}\right), \quad x \in \mathbb{R},$$

where $(\mathcal{F}_f(x))^{*l}$ is the l-fold convolution of $\mathcal{F}_f(\cdot)$ computed at the point x.

This distribution is a mixture of the discrete Poisson distribution with parameter 1 and a family of convolutions of $\mathcal{F}_f$'s, which is frequently referred to as the generalized Poisson distribution (see, e.g., [Lu], Ch. 12). Direct computation shows that the characteristic function $\phi_{\mathcal{K}'f}$ of $\mathcal{K}'f$ is given by

$$\begin{aligned}(\phi_{\mathcal{K}f}(t)) = \phi_{\mathcal{K}'f}(t) &= \exp\left(\int_{-\infty}^{\infty}(e^{itx}-1)d\mathcal{F}_f(x)\right)\\ &= \exp(\phi_f(t)-1) = \phi_{\pi(f)}(t), \quad t \in \mathbb{R}.\end{aligned}$$

This remark together with Definition 3.1 justifies the following assertion.

LEMMA 3.3: *If X is an r.i. space on $[0,1]$, then the operator $\mathcal{K}$ maps X boundedly into itself if and only if $X \in \mathbb{K}$.*

Remark 3.4: The statement of Lemma 3.3 in terms of operator boundedness will enable us later to apply interpolation techniques.

THEOREM 3.5: *Let $X \subseteq Y$ be r.i. spaces on $[0,1]$. Consider the following conditions:*

(i) *there exists $C > 0$ such that (1.2) holds for an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of independent random variables satisfying (1.1);*

(ii) *there exists $C > 0$ such that (1.2) holds for an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of independent identically distributed random variables satisfying (1.1);*

(iii) *the operator $\mathcal{K}$ acts boundedly from X into $Y^{\times\times}$;*

(iii)$'$ *the operator $\mathcal{K}$ acts boundedly from X into Y.*

The following implications hold: (iii)′$\Longrightarrow$(i)$\Longleftrightarrow$(ii). If the space Y is equipped with a Fatou norm, then (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii).

Proof: The implication (i)$\Longrightarrow$(ii) is obvious. Assume now that (ii) holds. Fix $f \in X$ and $n \in \mathbb{N}$ and choose $h \in X$ such that $\mathcal{F}_h = \mathcal{F}_f$ and such that h and $\chi_{[0,1/n]}$ are independent. Set $h_n := h\chi_{[0,1/n]}$, and let $\{\chi_{[0,1/n]}, h_{n,k}\}_{k=1}^n$ be a set of $(n+1)$ independent random variables such that $\mathcal{F}_{h_{n,k}} = \mathcal{F}_{h_n}$ for all $1 \le k \le n$. Since the functions $|\sum_{k=1}^n \bar{h}_{n,k}|$ and $|h|$ have the same distribution function, we conclude that the functions $|\sum_{k=1}^n \bar{h}_{n,k}|$ and $|f|$ are equidistributed, and therefore, by assumption,

$$\left\| \sum_{k=1}^n h_{n,k} \right\|_Y \le C \left\| \sum_{k=1}^n \bar{h}_{n,k} \right\|_X = C\|f\|_X. \tag{3.2}$$

A direct computation shows that $\phi_{h_n}(t) = n^{-1}\phi_f(t) + (1 - n^{-1})$ for all $t \in \mathbb{R}$. Hence, the characteristic function of the sum $H_n := \sum_{k=1}^n h_{n,k}$ is given by

$$\phi_{H_n}(t) = (n^{-1}(\phi_f(t) - 1) + 1)^n, \quad \forall t \in \mathbb{R}.$$

Since $\lim_{n\to\infty} \phi_{H_n}(t) = \exp(\phi_f(t) - 1) = \phi_{\pi(f)}(t)$, for all $t \in \mathbb{R}$ we see that H_n converges weakly to $\mathcal{K}f$. Combining this with (3.2), with [Br] Proposition 3, pp. 3–4 and with the fact that the natural embedding of Y into $Y^{\times\times}$ is an isometry, we conclude that $\|\mathcal{K}f\|_Y \le C\|f\|_X$. This completes the proof of the implication (ii)$\Longrightarrow$(iii).

Assume now that (iii) holds, i.e. that there exists $C < \infty$ such that $\|\mathcal{K}\|_{X\to Y^{\times\times}} \le C$. We shall first show the assertion (i) under an additional assumption that the sequence $\{f_k\}_{k=1}^n$ is symmetrically distributed. In [Pr], Yu. V. Prokhorov proved that in this case, if the sequence $\{h_k\}_{k=1}^n$ consists of independent random variables such that $\mathcal{F}_{h_k} = \mathcal{F}_{\pi(f_k)}$ for all $k = 1, 2, \ldots, n$, then

$$\lambda\left\{\left|\sum_{k=1}^n f_k\right| \ge x\right\} \le 8\lambda\left\{\left|\sum_{k=1}^n h_k\right| \ge \frac{x}{2}\right\}, \quad \forall x > 0.$$

It then follows from this inequality (see e.g. [KPS], *II*.(4.17)) that

$$\left\| \sum_{k=1}^n f_k \right\|_{Y^{\times\times}} \le 16 \left\| \sum_{k=1}^n h_k \right\|_{Y^{\times\times}}.$$

Setting now $f := \sum_{k=1}^n \bar{f}_k$ and taking into account that $\bar{f}_k\bar{f}_m = 0$ for all $k \ne m$, we have

$$\int_{-\infty}^{\infty} (e^{itx} - 1)d\mathcal{F}_f(x) = \sum_{k=1}^n \int_{-\infty}^{\infty} (e^{itx} - 1)d\mathcal{F}_{f_k}(x).$$

Therefore,

$$\phi_{\pi(f)}(t) = \prod_{k=1}^{n} \phi_{\pi(f_k)}(t) = \prod_{k=1}^{n} \phi_{h_k}(t), \quad t \in \mathbb{R}.$$

In other words, $\mathcal{F}_{\pi(f)} = \mathcal{F}_{\sum_{k=1}^n h_k}$ and hence, by the assumption, by the equality $\phi_{\pi(f)} = \phi_{\mathcal{K}(f)}$ (see the argument before Lemma 3.3) and by the inequality above,

$$\Big\| \sum_{k=1}^{n} f_k \Big\|_{Y^{\times\times}} \leq 16 \Big\| \sum_{k=1}^{n} h_k \Big\|_{Y^{\times\times}} = 16\|\mathcal{K}f\|_{Y^{\times\times}} \leq 16C\|f\|_X = 16C \Big\| \sum_{k=1}^{n} \bar{f}_k \Big\|_X .$$

Since $f_k \in X$, $k \geq 1$, $X \subset Y$ and Y has a Fatou norm, we conclude that $\|\sum_{k=1}^n f_k\|_Y \leq 16C\|\sum_{k=1}^n \bar{f}_k\|_X$, i.e. (i) holds. To complete the proof of Theorem 3.5, we need to consider the case when the sequence $\{f_k\}_{k=1}^n$ is no longer assumed to be symmetrically distributed.

Until the end of the proof we fix a number $a \in (0, 1/2]$ such that

$$\frac{\|\chi_{[0,a]}\|_Y}{\|\chi_{[0,1]}\|_Y} \leq \frac{1}{2}. \tag{3.3}$$

Combining the well known fact that every r.i. space X on $[0,1]$ contains L_∞ (see [KPS] Ch. 2, Section 4) with the assumption that $\mathcal{K}$ maps X into $Y^{\times\times}$, we infer that $\mathcal{K}\chi_{[0,1]} \in Y^{\times\times}$. It is easy to see that $\mathcal{K}\chi_{[0,1]} \notin L_\infty[0,1]$ (see detailed computations in Theorem 4.4 below). Therefore, $Y^{\times\times} \neq L_\infty[0,1]$ and hence $Y \neq L_\infty[0,1]$. The latter fact implies that the fundamental function of Y is continuous at 0 (see [KPS], Ch. 2), i.e. the inequality (3.3) is always satisfied for an appropriate choice of $a \leq 1/2$.

The remainder of the proof will be done in two steps. We first assume that the sequence $\{f_k\}_{k=1}^n$ is such that

$$\sum_{k=1}^{n} \lambda(\{f_k \neq 0\}) \leq a. \tag{3.4}$$

In this case, we shall use the standard "symmetrization" trick. Let $\{f'_k\}_{k=1}^n$ be a sequence of independent "copies" of the sequence $\{f_k\}_{k=1}^n$. We set $h_k := f_k - f'_k$ and let w_k: $\mathrm{supp}(h_k) \to E_k$ be measure preserving transformations, where E_k are pairwise disjoint subsets of $[0,1]$, $k = 1, 2, \ldots, n$ (note that we may choose E_k to be pairwise disjoint thanks to (3.4) and the assumption that $a \leq 1/2$). For $s \in E_k$ (respectively, $s \notin E_k$) and $k = 1, 2, \ldots, n$, we set

$$\bar{f}_k(s) := f_k(w_k^{-1}(s)), \quad \bar{f}'_k(s) := f'_k(w_k^{-1}(s)), \quad \bar{h}_k(s) := h_k(w_k^{-1}(s))$$

(respectively, $\bar{f}_k(s) = \bar{f}'_k(s) = \bar{h}_k(s) = 0$). Clearly, each of the sequences $\{\bar{f}_k\}_{k=1}^n$, $\{\bar{f}'_k\}_{k=1}^n$ and $\{\bar{h}_k\}_{k=1}^n$ consists of pairwise disjoint elements and $\mathcal{F}_{f_k} = \mathcal{F}_{\bar{f}_k}$, $\mathcal{F}_{f'_k} = \mathcal{F}_{\bar{f}'_k}$, $\mathcal{F}_{h_k} = \mathcal{F}_{\bar{h}_k}$ for all $k = 1, 2, \ldots, n$. It follows now from the symmetrization inequality (see [VTC], Ch. 5, Prop. 2.2) that

$$\lambda\{|h_k| > x\} \leq 2\lambda\{|f_k| > x/2\}, \quad x > 0.$$

Hence, it follows from (3.4) that, by the first part of the proof,

$$\left\|\sum_{k=1}^n h_k\right\|_Y \leq C\left\|\sum_{k=1}^n \bar{h}_k\right\|_X \leq 4C\left\|\sum_{k=1}^n \bar{f}_k\right\|_X. \tag{3.5}$$

On the other hand, we note first that by (3.4) we have $\|(\sum_{k=1}^n f_k)^*\chi_{[0,a]}\|_Y = \|\sum_{k=1}^n f_k\|_Y$. Now, writing

$$h_k = f_k - \mathbb{E}f_k - (f'_k - \mathbb{E}f_k) \quad \left(\mathbb{E}f := \int_0^1 f(x)dx\right)$$

and combining [Br], Prop. 11, p. 6 with (3.4), we get that for some constant $C(Y)$ we have

$$\begin{aligned}\left\|\sum_{k=1}^n h_k\right\|_Y &\geq C(Y)\left\|\sum_{k=1}^n f_k - \mathbb{E}\Big(\sum_{k=1}^n f_k\Big)\right\|_Y \\ &\geq C(Y)\left\|\sum_{k=1}^n f_k - \mathbb{E}\Big(\sum_{k=1}^n f_k\Big)\chi_{\mathrm{supp}(\sum_{k=1}^n f_k)}\right\|_Y \\ &\geq C(Y)\left(\left\|\sum_{k=1}^n f_k\right\|_Y - \mathbb{E}\left|\sum_{k=1}^n f_k\right|\|\chi_{[0,a]}\|_Y\right).\end{aligned}$$

Since the inequality $\|f\|_Y \geq \|\chi_{[0,1]}\|_Y\mathbb{E}|f|$ holds in every r.i. space Y (see [KPS] Ch. 2, Theorem 4.1), we infer from the inequality above and (3.3) that

$$\left\|\sum_{k=1}^n h_k\right\|_Y \geq C(Y)\left\|\sum_{k=1}^n f_k\right\|_Y\left(1 - \frac{\|\chi_{[0,a]}\|_Y}{\|\chi_{[0,1]}\|_Y}\right) \geq \frac{C(Y)}{2}\left\|\sum_{k=1}^n f_k\right\|_Y.$$

Together with (3.5) this yields (1.2) with the constant $8C/C(Y)$.

Finally, let us consider the setting of an arbitrary sequence of independent random variables $\{f_k\}_{k=1}^n \subset X$ satisfying condition (1.1). Without loss of generality, we may (and shall) assume that for any $b \in \mathbb{R}$ and each $k = 1, 2, \ldots, n$ we have $\lambda\{f_k = b\} = 0$. Since the sequence $\{f_k\}_{k=1}^n$ satisfies (1.1), we can select numbers $0 = b_0 < b_1 < b_2 < \cdots < b_l = \infty$, where $l := [1/a] + 1$ such that

$$\sum_{i=1}^n \lambda\{b_{k-1} < f_i < b_k\} \leq a, \quad \forall k = 1, 2, \ldots, l.$$

Set $f_{i,k}(x) := f_i(x)\chi_{\{b_{k-1}<f_i<b_k\}}(x)$, $i = 1,2,\dots,n$, $k = 1,2,\dots,l$, $x \in [0,1]$. It is clear that $f_i = \sum_{k=1}^{l} f_{i,k}$ and that for every $k = 1,2,\dots,l$ the sequence $\{f_{i,k}\}_{i=1}^n \subset X$ satisfies condition (3.4). Hence, using the first step of the proof and the fact that $|\bar{f}_{i,k}| \le |\bar{f}_i|$ ($k = 1,2,\dots,l$), we obtain

$$\Big\|\sum_{i=1}^n f_i\Big\|_Y \le \sum_{k=1}^l \Big\|\sum_{i=1}^n f_{i,k}\Big\|_Y \le C\sum_{k=1}^{l} \Big\|\sum_{i=1}^n \bar{f}_{i,k}\Big\|_X \le C\Big(\Big[\frac{1}{a}\Big]+1\Big)\Big\|\sum_{i=1}^n \bar{f}_i\Big\|_X.$$

The proof of the implication (iii)$'$ $\Longrightarrow$(i) follows along the same lines as the proof of the implication (iii)$\Longrightarrow$(i). This completes the proof of Theorem 3.5. ∎

COROLLARY 3.6 ([Br], Lemma 4, p. 13): *If an r.i. space $X \in \mathbb{K}$, then there exists $C > 0$ such that for every sequence $\{f_k\}_{k=1}^n \subset X$ of independent random variables satisfying (1.1) we have*

$$\Big\|\sum_{k=1}^n f_k\Big\|_X \le C\Big\|\sum_{k=1}^n \bar{f}_k\Big\|_X.$$

4. The operator $\mathcal{K}$ in exponential Orlicz spaces $\exp(L_p)$, $0 < p \le \infty$

Let Φ be an Orlicz function on $[0,\infty)$, that is, Φ is a continuous convex increasing function on $[0,\infty)$ satisfying $\Phi(0) = 0$ and $\Phi(\infty) = \infty$. The Orlicz space $L_\Phi = L_\Phi[0,\alpha)$, $0 < \alpha \le \infty$ is the space of all Lebesgue measurable functions f on $[0,\alpha)$ for which

$$\int_0^\alpha \Phi\Big(\frac{|f(t)|}{\rho}\Big)dt < \infty$$

for some $\rho > 0$. The (Luxemburg) norm in $L_\Phi = L_\Phi[0,\alpha)$ is defined by

$$\|f\|_\Phi = \inf\Big\{\rho > 0\colon \int_0^\alpha \Phi\Big(\frac{|f(t)|}{\rho}\Big)dt \le 1\Big\}.$$

The Orlicz space $L_\Phi[0,\alpha)$ is maximal. In the case that $\alpha < \infty$, $L_\Phi[0,\alpha)$ is separable if and only if Φ satisfies the Δ_2-condition at ∞. The space $L_\Phi[0,\infty)$ is separable if and only if Φ satisfies the Δ_2-condition both at ∞ and at 0 (see, for example, [BS], [LT]).

It is easy to see that for every $p \in (0,\infty)$, the function

$$N_p(t) := e^{|t|^p} - \sum_{k=0}^{[1/p]} \frac{|t|^{kp}}{k!}, \quad p \in (0,1); \quad N_p(t) := e^{|t|^p} - 1, \quad p \ge 1, \quad t \in \mathbb{R},$$

is an Orlicz function (here, as usual, $[1/p]$ denotes the integral part of $1/p$). The corresponding Orlicz space, L_{N_p}, is frequently denoted by $\exp(L_p)$ ($\exp(L_\infty) := L_\infty$). It follows from the original paper of Kruglov [K] that $\exp(L_p) \in \mathbb{K}$ for all $0 < p \le 1$ and it is noted in [Br] that this is no longer the case when $p > 1$. For completeness sake, we present a simple proof of Kruglov's result.

PROPOSITION 4.1: *If Φ is an Orlicz function such that for some constant $B \ge 1$,*

$$\Phi(x+y) \le B\Phi(x)\Phi(y), \quad \forall x, y > 0,$$

then $\mathcal{K}$ (boundedly) maps L_Φ into itself.

Proof: We shall denote $\int_0^1 f(x)dx$ by $\mathbb{E}f$. Let $f = f_1 \in L_\Phi$ with $\mathbb{E}(\Phi|f|) := \int_0^1 \Phi(|f(x)|)dx \le 1$ (i.e. $\|f\|_{L_\Phi} \le 1$) and let $\{f_k\}_{k=1}^\infty$ be a sequence of independent identically distributed random variables. For every $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathbb{E}\Big(\Phi|\sum_{k=1}^n f_k|\Big) &\le B\mathbb{E}\Big(\Phi|\sum_{k=1}^{n-1} f_k|\Phi|f_n|\Big)\\ &= B\mathbb{E}\Big(\Phi|\sum_{k=1}^{n-1} f_k|\Big)\mathbb{E}(\Phi|f_n|) \le \cdots \le B^{n-1}.\end{aligned}$$

Using the definition of the operator $\mathcal{K}$, we obtain

$$\begin{aligned}\mathbb{E}(\Phi|\mathcal{K}f|) &= \sum_{n=1}^\infty \mathbb{E}\Big(\Phi\Big|\sum_{k=1}^n f_{n,k}\chi_{E_n}\Big|\Big)\\ &= \sum_{n=1}^\infty \mathbb{E}\Big(\Phi\Big|\sum_{k=1}^n f_{n,k}\Big|\mathbb{E}(\Phi(\chi_{E_n}))\Big)\\ &\le \Phi(1)\sum_{n=1}^\infty B^{n-1}\lambda(E_n) = \Phi(1)\frac{1}{e}\sum_{n=1}^\infty \frac{B^{n-1}}{n!}\\ &= \frac{\Phi(1)(e^B-1)}{eB}. \qquad \blacksquare\end{aligned}$$

Combining Proposition 4.1, Lemma 3.3 and Theorem 3.5 we obtain the following corollary.

COROLLARY 4.2: *If Φ satisfies the conditions of Proposition 4.1, then*

(i) (see [K]) *the Orlicz space $L_\Phi \in \mathbb{K}$;*

(ii) *there exists $C > 0$ such that for an arbitrary sequence $\{f_k\}_{k=1}^n \subset L_\Phi$ of independent random variables satisfying (1.1), the following inequality holds:*

$$\Big\|\sum_{k=1}^n f_k\Big\|_{L_\Phi} \le C\Big\|\sum_{k=1}^n \bar{f}_k\Big\|_{L_\Phi}.$$

We remark that the assertion of Proposition 4.1 (and thus that of Corollary 4.2) remain valid (with analogous proof) if the Orlicz function Φ satisfies the following condition:

$$\Phi(x+y) \le B(\Phi(x)+\Phi(y)), \quad \forall x, y > 0.$$

We introduce next the Orlicz functions

$$M_p(t) := e^{|t|\ln^{1/p}(e+|t|)} - 1, \quad p > 0, \quad t \in \mathbb{R}.$$

Denote by Ψ the set of increasing concave functions on $[0,\infty)$ with $\psi(0) = \psi(+0) = 0$. If $\psi \in \Psi$, then the Marcinkiewicz space $M_\psi[0,\alpha)$ consists of those measurable functions x for which

$$\|x\|_{M_\psi[0,\alpha)} := \sup_{0<s<\alpha} \frac{1}{\psi(s)} \int_0^s x^*(t)dt < \infty.$$

It is useful to note that for every $p > 0$, the Orlicz space L_{M_p} (respectively, L_{N_p}) coincides with the Marcinkiewicz space M_{ψ_p} (respectively, M_{ϕ_p}), where

$$\psi_p(t) := \frac{t\ln(e/t)}{\ln^{1/p}(\ln(e^e/t))} \quad (\text{respectively}, \phi_p(t) := t\ln^{1/p}(e/t)), \quad t \ge 0.$$

For the reader's convenience, we include a short proof of this observation below. In this proof, as well as in further arguments in this section, we shall frequently use the equivalent expressions for the norms on the Marcinkiewicz spaces M_{ψ_p} and M_{ϕ_p} on the interval $[0,1]$ which follow from [KPS] Theorem II.5.3,

$$(4.1)\qquad \begin{aligned} \|x\|_{M_{\psi_p}} &\asymp \sup_{t\in(0,1)} \frac{t}{\psi_p(t)} x^*(t), \quad x \in M_{\psi_p} \quad \text{and} \\ \|x\|_{M_{\phi_p}} &\asymp \sup_{t\in(0,1)} \frac{t}{\phi_p(t)} x^*(t), \quad x \in M_{\phi_p}. \end{aligned}$$

LEMMA 4.3: *The equalities $L_{M_p} = M_{\psi_p}$ and* $(\exp(L_p) =)L_{N_p} = M_{\phi_p}$ *(with norm equivalence) hold for every $p > 0$.*

Proof: We shall prove only the first equality, since the proof of the second is similar (and simpler). Fix $p > 0$. It is sufficient to prove that the fundamental functions $\phi_{L_{M_p}}$ and $\phi_{M_{\psi_p}}$ are equivalent and that the function f_p: $t \to \psi_p(t)/t$ belongs to the Orlicz space L_{M_p} (see [KPS], Section II.5.4 and (4.1) above). Since

$$\phi_{L_{M_p}}(t) = \frac{1}{M_p^{-1}(1/t)}, \quad \phi_{M_{\psi_p}}(t) = \frac{\ln^{1/p}(\ln(e^e/t))}{\ln(e/t)} \ \Big(= \frac{1}{f_p(t)}\Big), \quad t > 0$$

(see [BS], Lemma 4.8.17 and [KPS], Theorem II.5.7), it suffices to check that the functions $M_p^{-1}(1/\cdot)$ and $f_p(\cdot)$ are equivalent in a neighbourhood of 0. Since for every positive c we have

$$\lim_{t\to 0}\frac{\ln^{1/p}(e+c\cdot\frac{\ln(e/t)}{\ln^{1/p}(\ln(e^e/t))})}{\ln^{1/p}(\ln(e^e/t))}=1,$$

we can select $\delta>0$ such that for all $t\in(0,\delta)$ we have

$$M_p\Big(\frac{1}{2}f_p(t)\Big)\le e^{\frac{2}{3}\ln(e/t)}\le 1/t\le e^{\frac{3}{2}\ln(e/t)}\le M_p(2f_p(t)). \tag{4.2}$$

It follows from (4.2) that functions $M_p^{-1}(1/\cdot)$ and $f_p(\cdot)$ are equivalent in $(0,\delta)$. Finally, the embedding $f_p\in L_{M_p}$ follows immediately from the first inequality on the left in (4.2). ∎

It follows from Proposition 4.1 that for all $p\in(0,1]$ the operator $\mathcal{K}$ acts boundedly on $\exp(L_p)$. To describe the behaviour of the operator $\mathcal{K}$ on the spaces $\exp(L_p)$ for $p\in(1,\infty)$ we set, for brevity,

$$\begin{aligned}\mathcal{Y}_p:=\{&\text{the set of all r.i. spaces } Y \text{ such that}\\ &\mathcal{K} \text{ maps } \exp(L_p) \text{ boundedly into } Y\},\quad p\in(1,\infty]\end{aligned}$$

(with the understanding that $\exp(L_\infty):=L_\infty$).

The following result shows that the space L_{M_1} plays a crucial role in the study of series of uniformly bounded independent random variables (see also [KW], Corollary 3.5.2 for a somewhat related result).

THEOREM 4.4: *The set $\mathcal{Y}_\infty$ ordered by inclusion has a unique minimal element, L_{M_1}.*

Proof: Let $g(x)=\mathcal{K}\chi_{[0,1]}(x)$. It follows from the definition of the operator $\mathcal{K}$ (see also (3.1)′) that

$$g^*(t)=k,\quad \forall t\in(t_k,t_{k-1}),\quad t_0=1,\quad t_k:=\frac{1}{e}\sum_{i=k}^{\infty}\frac{1}{i!},\quad k\in N.$$

Since L_{M_1} coincides with the Marcinkiewicz space M_{ψ_1}, it is sufficient to show (see (4.1) and the proof of Lemma 4.3) that $g^*(\cdot)$ and $f(\cdot):=\ln(e/\cdot)/\ln(\ln(e^e/\cdot))$ ($=f_1$ from the proof of Lemma 4.3) are equivalent in a neighborhood of 0. Since the function f is decreasing on $(0,1)$ and since the function $g^*\equiv k$ on every interval (t_{k-1},t_k), $k\in\mathbb{N}$, it is sufficient to show that

$$\frac{1}{2}\le\lim_{k\to\infty}\frac{f(t_{k-1})}{k}\le\lim_{k\to\infty}\frac{f(t_k)}{k}\le 1.$$

To this end, noting that for all $k \in \mathbb{N}$

$$\begin{aligned} k!t_k &= \frac{1}{e}\Big(1+\sum_{i=1}^{\infty}\frac{1}{(k+1)(k+2)\cdots(k+i)}\Big) \\ &\le \frac{1}{e}\Big(1+\frac{1}{k+1}+\sum_{i=1}^{\infty}\frac{1}{(k+i)(k+i+1)}\Big) \\ &\le \frac{1}{e}\Big(1+\frac{2}{k+1}\Big) \le \frac{2}{e}, \end{aligned}$$

and thus

$$\frac{1}{e}\frac{1}{k!} \le t_k \le \frac{2}{e}\frac{1}{k!}, \quad k \in \mathbb{N}.$$

Combining Stirling's formula $k! \backsim \sqrt{2\pi k}k^k e^{-k}$ with the inequality above, it follows that for all sufficiently large k, we have

$$k^{-k} \le t_k < t_{k-1} \le (k-1)^{-\frac{1}{2}k}.$$

Thus,

$$\frac{1}{2} = \lim_{k\to\infty}\frac{f((k-1)^{-\frac{1}{2}k})}{k} \le \lim_{k\to\infty}\frac{f(t_{k-1})}{k} \le \lim_{k\to\infty}\frac{f(t_k)}{k} \le \lim_{k\to\infty}\frac{f(k^{-k})}{k} = 1. \qquad \blacksquare$$

Remark: An alternative proof of Theorem 4.4 may be obtained via the following refined version of the Rosenthal Inequality from [La],

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_{L_p} \le C\frac{p}{\ln(p+1)}\Big\|\sum_{k=1}^{n}\bar{f}_k\Big\|_{L_p}.$$

Here $\{f_k\}_{k=1}^n$ is an arbitrary sequence of independent random variables in $L_\infty(0,1)$ satisfying condition (1.1) and C does not depend on $p \ge 1$. Indeed, the inequality above implies that

$$\sup_{p\ge1}\Big\{\frac{\ln(p+1)}{p}\Big\|\sum_{k=1}^{n} f_k\Big\|_{L_p}\Big\} \le C\Big\|\sum_{k=1}^{n}\bar{f}_k\Big\|_{\infty}.$$

Using the Taylor decomposition of the function $M_1(t) = (t+1)^t - 1$ at the neighbourhood of 0, it is not hard to see that the left hand side of the preceding inequality is equivalent to $\|\sum_{k=1}^n f_k\|_{L_{M_1}}$ (see also [A]).

Using Corollary 4.2, we shall now show how the result of Theorem 4.4 can be extended to all values of $p \in (1,\infty)$.

THEOREM 4.5: *The set $\mathcal{Y}_p$, $1 < p < \infty$ ordered by inclusion has a unique minimal element, L_{M_q}, where $1/p + 1/q = 1$.*

Proof: It follows from Proposition 4.1 and Theorem 4.4 that the operator $\mathcal{K}$ maps boundedly L_∞ into L_{M_1} and $\exp(L_1)$ ($= L_{N_1}$) into itself. Hence, using the real method of interpolation (see, e.g., [LT]), we infer that

$$\mathcal{K}\colon (L_\infty, L_{N_1})_{\theta,\infty} \to (L_{M_1}, L_{N_1})_{\theta,\infty}, \quad 0 < \theta < 1.$$

By Lemma 4.3, $L_{N_1} = M_{\phi_1}$, $L_{M_1} = M_{\psi_1}$ and it is well known that the space L_∞ is the Marcinkiewicz space M_{id}, where $id(t) = t$ for all $t \geq 0$. Therefore, using the known description of the spaces obtained by the real method of interpolation in Marcinkiewicz couples (see, e.g., [O], Ex. 7.1.3, p. 428) we obtain

$$(L_\infty, L_{N_1})_{\theta,\infty} = L_{N_{\theta^{-1}}}, \quad (L_{M_1}, L_{N_1})_{\theta,\infty} = L_{M_{(1-\theta)^{-1}}}.$$

Setting $p = \theta^{-1}$, we immediately infer that $\mathcal{K}$ maps L_{N_p} into L_{M_q}. It now follows from (4.1) that in order to complete the proof of Theorem 4.5, it is sufficient to show that for some scalar C (possibly dependent on p) and all sufficiently small $t > 0$ we have

$$h_0(t) \leq C(\mathcal{K}g_0)^*(t),$$

where

$$g_0(\cdot) := \ln^{1/p}(e/\cdot) \quad \text{and} \quad h_0(\cdot) := \frac{\ln(e/\cdot)}{\ln^{1/q}(\ln(e^e/\cdot))}.$$

Since $h_0 = h_0^*$, the latter inequality holds if and only if for all sufficiently large $\tau > 0$ we have

$$\lambda\{\mathcal{K}g_0 > \tau/C\} \geq \lambda\{h_0 > \tau\}.$$

Further, a direct verification shows that $\lambda\{h_0 > \tau\} \leq e^{-\tau \ln^{1/q}\tau}$ for all sufficiently large τ, hence it is sufficient to prove that $\lambda\{\mathcal{K}g_0 > \tau/3\} \geq e^{-\tau\ln^{1/q}\tau}$, or equivalently,

$$\lambda\{\mathcal{K}g_0 > \tau\} \geq e^{-3\tau\ln^{1/q}\tau} \tag{4.3}$$

for all sufficiently large τ. We have

$$\lambda\{g_0 > \tau\} = \exp(1 - \tau^{-p}) \geq e^{-\tau^p}.$$

Fix $n \in \mathbb{N}$ and let $g_1, g_2, \ldots, g_n$ be independent copies of g_0. Setting $\mathbf{g} := \min(g_1, g_2, \ldots, g_n)$ we have

$$\lambda\{\mathbf{g} > \tau\} = \lambda(\cap_{i=1}^n \{g_i > \tau\}) = \prod_{i=1}^n \lambda\{g_i > \tau\} \geq e^{-n\tau^p}.$$

Hence, using the definition of the operator $\mathcal{K}$, we obtain

$$\begin{aligned}\lambda\{\mathcal{K}g_0 > \tau\} &= \lambda\Big\{\sum_{n=1}^{\infty}\sum_{k=1}^{n} g_{n,k}\chi_{E_n} > \tau\Big\}\\ &\geq \lambda\Big\{\sum_{n=1}^{\infty} n\min\{g_{n,1},\dots,g_{n,n}\}\chi_{E_n} > \tau\Big\}\\ &= \sum_{n=1}^{\infty}\lambda\{\min_{k=1,2,\dots,n} g_{n,k} > \tau/n\}\lambda(E_n)\\ &\geq \sum_{n=1}^{\infty} e^{-n(\tau/n)^p}\lambda(E_n) = \frac{1}{e}\sum_{n=1}^{\infty} e^{-n^{1-p}\tau^p}\frac{1}{n!}.\end{aligned}$$

Note that if $n \geq \tau \ln^{-1/p}(1+\tau)$, then (equivalently) $\tau\ln^{1-1/p}(1+\tau) \geq n^{1-p}\tau^p$. Therefore,

$$\begin{aligned}\lambda\{\mathcal{K}g_0 > \tau\} &\geq \exp(-\tau\ln^{1-1/p}(1+\tau))\frac{1}{e}\sum_{n\geq\tau\ln^{-1/p}(1+\tau)}\frac{1}{n!}\\ &\geq \exp(-\tau\ln^{1-1/p}\tau)\frac{1}{e[2\tau\ln^{-1/p}\tau]!}.\end{aligned}\tag{4.4}$$

Using Stirling's formula, we see that for sufficiently large τ

$$\frac{1}{e[2\tau\ln^{-1/p}\tau]!} \geq \exp(-2\tau\ln^{-1/p}\tau\ln(2\tau\ln^{-1/p}\tau)) \geq e^{-2\tau\ln^{1-1/p}\tau}.$$

Combining this estimate with (4.4), we obtain (4.3). This completes the proof of Theorem 4.5. ∎

Corollary 4.6: *Let $p \in (1,\infty]$ and $q = p/(p-1)$. There exists $C_p > 0$ such that for any finite sequence of independent random variables $\{f_k\}_{k=1}^n \subset \exp(L_p)$ satisfying (1.1), we have*

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_{L^0_{M_q}} \leq C_p\Big\|\sum_{k=1}^{n}\bar{f}_k\Big\|_{\exp(L_p)}.$$

Moreover, if an r.i. space Y with a Fatou norm is such that for all sequences $\{f_k\}_{k=1}^n \subset \exp(L_p)$ as above we have

$$\Big\|\sum_{k=1}^{n} f_k\Big\|_{Y} \leq C\Big\|\sum_{k=1}^{n}\bar{f}_k\Big\|_{\exp(L_p)},$$

then necessarily, $L^0_{M_q} \subseteq Y$.

Proof: If $p = \infty$ (respectively, $p < \infty$), then combine Theorems 3.5 and 4.4 (respectively, 4.5) and use the fact that $L^0_{M_q}$ is the smallest among all r.i. spaces Y satisfying $Y^{\times\times} \supseteq L_{M_q}$. ∎

5. The operator $\mathcal{K}$ in Lorentz spaces

Recall that Ψ is the set of all increasing concave functions on $[0,\infty)$ with $\psi(0) = \psi(+0) = 0$. If $\psi \in \Psi$, then the Lorentz space $(\Lambda_\psi[0,\alpha), \|\cdot\|_{\Lambda_\psi[0,\alpha)})$ on $[0,\alpha)$ is the space of all measurable functions x on $[0,\alpha)$ for which

$$\|x\|_{\Lambda_\psi[0,\alpha)} := \int_0^\alpha x^*(t)d\psi(t) < \infty.$$

The space $\Lambda_\psi[0,\alpha)$, $1 \le \alpha < \infty$ is always separable.

The following theorem describes Lorentz spaces Λ_ψ on which the operator $\mathcal{K}$ acts boundedly in terms of the function $\psi \in \Psi$.

THEOREM 5.1: *Let $\psi \in \Psi$. The operator $\mathcal{K}$ maps the Lorentz space Λ_ψ into itself (i.e. $\Lambda_\psi \in \mathbb{K}$) if and only if there exists $C > 0$ such that*

$$\sum_{k=1}^{\infty} \psi\Big(\frac{u^k}{k!}\Big) \le C\psi(u), \quad u \in (0,1]. \tag{5.1}$$

Proof: Let $f := \chi_{(0,u]}$, $u \in (0,1]$ and let the sequences $\{f_{n,k}\}_{k=1}^n$, $\{\chi_{E_n}\}_{n=1}^\infty$, $n \in \mathbb{N}$ satisfy the conditions (i) and (ii) from $(3.1)'$. Then, the function $f_n := \sum_{k=1}^n f_{n,k}$ has a binomial distribution with parameter u, i.e.

$$\lambda\{f_n = k\} = C_n^k u^k (1-u)^{n-k}, \quad k = 0,1,\dots,n, \quad n \in \mathbb{N},$$

where $C_n^k = n!/k!(n-k)!$. Therefore, we have (see $(3.1)'$)

$$\mathcal{K}f(s) = \sum_{k=1}^{\infty} k\chi_{A_k}(s),$$

where

$$\begin{aligned}\lambda(A_k) &= \sum_{n=k}^{\infty} C_n^k u^k (1-u)^{n-k}\lambda(E_n) = \frac{1}{e}\sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!}u^k(1-u)^{n-k}\frac{1}{n!} \\ &= \frac{1}{e}\frac{u^k}{(1-u)^k k!}\sum_{n=k}^{\infty}\frac{(1-u)^n}{(n-k)!} = \frac{1}{e}\frac{u^k(1-u)^k}{(1-u)^k k!}\sum_{n=0}^{\infty}\frac{(1-u)^n}{n!} \\ &= e^{-u}\frac{u^k}{k!}.\end{aligned}$$

Hence, the function $\mathcal{K}f$ has Poisson distribution with parameter u, which coincides with the distribution of the function

$$h(s) := \sum_{k=1}^{\infty} \chi_{(0,\tau_k]}(s), \quad s \in [0,1],$$

where $\tau_k := e^{-u}\sum_{i=k}^{\infty}(u^i/i!)$, $k \in \mathbb{N}$. Therefore,

$$\|\mathcal{K}f\|_{\Lambda_\psi} = \|h\|_{\Lambda_\psi} = \sum_{k=1}^{\infty}\int_0^{\tau_k} d\psi(s) = \sum_{k=1}^{\infty}\psi(\tau_k). \tag{5.2}$$

Since $\mathcal{K}$ is a bounded positive operator from Λ_ψ into the space $S[0,1]$, and since the extreme points of the positive part of the unit sphere of the space Λ_ψ are given by all normalized indicator functions of measurable subsets of $[0,1]$, it is sufficient to verify the boundedness of $\mathcal{K}$ on Λ_ψ on the set of all such functions (see Corollary 1 to Lemma 5.2 in [KPS], Chapter *II*.5). Therefore, it follows from (5.2) that $\mathcal{K}$ acts boundedly on the space Λ_ψ if and only if there exists $C > 0$ such that

$$\sum_{k=1}^{\infty}\psi(\tau_k) \le C\psi(u), \quad \forall u \in (0,1]. \tag{5.3}$$

It is easy to show (see a similar argument in the proof of Theorem 4.4) that

$$\frac{1}{e}\frac{u^k}{k!} \le \tau_k \le 2\frac{u^k}{k!}, \quad \forall k \in \mathbb{N}.$$

Combining this inequality with the fact that ψ is a concave function, we infer that (5.3) and (5.1) are equivalent conditions. ∎

Remark 5.2: Arguing in a similar fashion, it can be shown that for a given $\psi \in \Psi$ the operator $\mathcal{K}$ boundedly maps Λ_ψ into $M_{t/\psi(t)}$ (or, equivalently, using customary language of interpolation theory [KPS], [BS] the operator $\mathcal{K}$ is of weak type (ψ,ψ)) if and only if

$$\sup_{u\in(0,1],k\in\mathbb{N}} \frac{k\psi(u^k/k!)}{\psi(u)} < \infty. \tag{5.4}$$

Indeed, let $u, \tau_k, k \in \mathbb{N}$ and functions f, $h = \mathcal{K}f$ be as in the proof of Theorem 5.1. It is not difficult to see that we may compute the norm of h in the Marcinkiewicz space $M_{t/\psi(t)}$ as follows:

$$\begin{aligned}\|h\|_{M_{t/\psi(t)}} &= \sup_{k\in\mathbb{N}} \frac{\psi(\tau_k)\int_0^{\tau_k}(\sum_{i=1}^{\infty}\chi_{(0,\tau_i)}(s))ds}{\tau_k} \\ &= \sup_{k\in\mathbb{N}} \frac{\psi(\tau_k)[(k-1)\tau_k + \sum_{i=k}^{\infty}\tau_i]}{\tau_k} \asymp \sup_{k\in\mathbb{N}}(k\psi(\tau_k)).\end{aligned}$$

The estimate (5.4) now follows exactly as in the proof of Theorem 5.1.

The condition (5.1) for concave functions $\psi \in \Psi$ appears also in [MS] (see Eq. (23) there), which studies random rearrangements in r.i. spaces. The following technical estimate is established in [MS], Lemma 11.

LEMMA 5.3 ([MS]): *If $\psi \in \Psi$ is such that $\psi(1) = 1$ and $\psi(t) \le at^{1/p}$ for all $t \in [0,1]$ and some $p, a \in [1,\infty)$, then*

$$\sup_{0<u\le 1} \frac{1}{\psi(u)} \sum_{k=1}^{\infty} \psi\Big(\frac{u^k}{k!}\Big) \le 5ap. \tag{5.5}$$

A special case of the following result is given in [Br], Theorem 2, p. 16.

COROLLARY 5.4: *If an r.i. space X with a Fatou norm contains $L_p[0,1]$ for some $p \in [1,\infty)$, then the operator $\mathcal{K}$ acts boundedly from X into $X^{\times\times}$. In particular, there exists $C > 0$ such that for an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of independent random variables satisfying (1.1) we have*

$$\Big\| \sum_{k=1}^n f_k \Big\|_X \le C \Big\| \sum_{k=1}^n \bar{f}_k \Big\|_X. \tag{5.6}$$

Proof: (i) By the definition of $X^{\times\times}$, we have for every $x \in X^{\times\times}$

$$\|x\|_{X^{\times\times}} := \sup\Big\{ \int_0^1 |x(t)y(t)|dt \colon y \in X^{\times}, \|y\|_{X^{\times}} \le 1 \Big\}.$$

This can also be interpreted as

$$X^{\times\times} = \cap \Lambda_{\psi_y}, \quad \|x\|_{X^{\times\times}} = \sup \|x\|_{\Lambda_{\psi_y}}, \tag{5.7}$$

where the intersection and supremum are taken over all $\psi_y \in \Psi$ such that

$$\psi_y(t) = \int_0^t y^*(s)ds, \quad t \in [0,1], \quad y \in X^{\times}, \quad \|y\|_{X^{\times}} \le 1.$$

Following [MS], for every such $\psi_y \in \Psi$ we set

$$\theta_y(t) := \frac{\psi_y(t) + t}{\|y\|_1 + 1} \quad (y \in X^{\times}, t \in [0,1]).$$

Clearly, $\theta_y \in \Psi$ and $\theta_y(1) = 1$ for all $y \in X^{\times}$. Since (see [KPS], $(II.4.6)$)

$$\|y\|_1 + 1 \le \frac{1}{\phi_{X^{\times}}(1)} \|y\|_{X^{\times}} + 1 \le \phi_X(1) + 1, \quad y \in X^{\times}, \quad \|y\|_{X^{\times}} \le 1,$$

we deduce that

$$(5.8)\qquad \|x\|_{\Lambda_{\psi_y}} \le (\phi_X(1)+1)\|x\|_{\Lambda_{\theta_y}}, \quad x \in X^{\times\times}, y \in X^{\times}, \quad \|y\|_{X^\times} \le 1.$$

On the other hand, for all x's as above, it follows from (5.7) that

$$\|x\|_{\Lambda_{\theta_y}} \le \|x\|_{\Lambda_{\psi_y}} + \|x\|_1 \le \|x\|_{\Lambda_{\psi_y}} + \frac{1}{\phi_{X^{\times\times}}(1)}\|x\|_{X^{\times\times}} \le (1+\frac{1}{\phi_X(1)})\|x\|_{X^{\times\times}}.$$

Combining this estimate with (5.7) and (5.8), we see that

$$X^{\times\times} = \cap\Lambda_{\theta_y}, \quad \|x\|_{X^{\times\times}} \asymp \sup_{y\in X^\times, \|y\|_{X^\times}\le 1} \|x\|_{\Lambda_{\theta_y}},$$

where the intersection is taken over all θ_y's as above. Further, it follows from the assumption $L_p \subset X$ that $X^\times \subset L_p^\times = L_q$ (here $q = \frac{p}{p-1}$). Therefore, again using [KPS], $(II.4.6)$, we arrive at

$$\theta_y(t) \le \int_0^t y^*(s)ds + t \le \|y\|_q t^{1/p} + t \le a\|y\|_{X^\times} t^{1/p} + t \le (a+1)t^{1/p}.$$

It follows now, from Lemma 5.3 and from Theorem 5.1 (and its proof), that $\mathcal{K}$ maps Λ_{θ_y} into itself for every $y \in X^\times$ with $\|y\|_{X^\times} \le 1$, and moreover

$$\sup_{\|y\|_{X^\times}\le 1} \|\mathcal{K}\|_{\Lambda_{\theta_y}\to\Lambda_{\theta_y}} = C < \infty.$$

Thus for every $x \in X^{\times\times}$ we have

$$\|\mathcal{K}x\|_{X^{\times\times}} \asymp \sup_{\|y\|_{X^\times}\le 1} \|\mathcal{K}x\|_{\Lambda_{\theta_y}} \le C \sup_{\|y\|_{X^\times}\le 1} \|x\|_{\Lambda_{\theta_y}} \asymp \|x\|_{X^{\times\times}}.$$

Since X is equipped with a Fatou norm, the result now follows from Theorem 3.5. ∎

The proof of the following assertion may be obtained in the same way as in the proof of Theorem 14 [MS] and is therefore omitted.

LEMMA 5.5: *If $\phi \in \Psi$, $\phi(1) = 1$ is such that for any $\psi \in \Psi$ with $\psi(1) = 1$ and $\psi \le \phi$ condition (5.4) holds, then necessarily $\phi(t) \le at^\alpha$ for some $a \ge 1$, $\alpha \in (0,1]$ and all $t \in [0,1]$.*

Using Remark 5.2 and Lemma 5.5, it is now possible to prove the converse, in a certain sense, to the main result of [JS] (for normed r.i. spaces).

COROLLARY 5.6: *If an r.i. space E is such that for every maximal r.i. space $X \supset E$ there exists $C > 0$ such that (5.6) holds for an arbitrary sequence $\{f_k\}_{k=1}^n \subset X$ of independent random variables satisfying (1.1), then E contains an L_p-space for some $p \in [1, \infty)$.*

Proof: According to [KPS], Theorem II.5.7, $E \subseteq M_{t/\phi_E}$, where ϕ_E is the fundamental function of E. Therefore, for every function $\psi \in \Psi$ such that $\psi \leq C\phi_E$ for some constant $C > 0$, we have $E \subseteq M_{t/\phi_E(t)} \subseteq M_{t/\psi(t)}$. Hence, by the assumption and Theorem 3.5, the operator $\mathcal{K}$ boundedly maps $M_{t/\psi(t)}$ into itself and this implies (see Remark 5.2) that condition (5.4) holds for every function ψ as above. By Lemma 5.5 this guarantees that $\phi_E(t) \leq \phi_E(1)at^\alpha$ for some $a \geq 1$, $\alpha \in (0, 1]$ and all $t \in [0, 1]$. The latter fact and [KPS], Theorem II.5.5 guarantee that $E \supseteq \Lambda_{\phi_E} \supseteq \Lambda_{t^\alpha}$ and since Λ_{t^α} contains $L_p[0, 1]$, $p > 1/\alpha$, we are done. ∎

COROLLARY 5.7: *If $\psi \in \Psi$ is such that for every $\alpha \in (0, 1]$*

$$\sup_{t\in(0,1]} t^{-\alpha}\psi(t) = \infty,$$

then there exists $\phi \in \Psi$ such that $\phi \leq C\psi$ for some $C > 0$ such that the operator $\mathcal{K}$ is not bounded on any r.i. space X with the fundamental function $\phi_X = \phi$.

The preceding results show that it is not possible to answer the main question (see Section 1) in terms analogous to the Rodin–Semenov characterization of r.i. spaces satisfying the Khintchine Inequality. It should be also mentioned that there are many r.i. spaces which have the Kruglov property and which do not necessarily contain some L_p-space, $1 < p < \infty$ (see, e.g., Corollary 4.2). The next proposition presents a general method to exhibit such examples among Marcinkiewicz spaces.

PROPOSITION 5.8: *For every quasi-concave non-negative function ρ on $[0, 1]$, let $\psi(t) := \rho(t \ln e/t)$. If $\tilde{\psi}(t) := t/\psi(t)$, $t \in [0, 1]$, then $M_{\tilde{\psi}} \in \mathbb{K}$.*

Proof: We denote by $l_\infty(1/\rho)$ the space of all two-sided scalar sequences $\mathbf{a} = \{a_k\}_{k=-\infty}^{\infty}$ such that

$$\|\mathbf{a}\|_{l_\infty(1/\rho)} := \left\| \left\{ \frac{a_k}{\rho(2^k)} \right\}_{k=-\infty}^{\infty} \right\|_{l_\infty} < \infty.$$

Since, by Proposition 4.1, the operator $\mathcal{K}$ is bounded on L_1 and L_{N_1}, we infer that it is also bounded on the space $(L_1, L_{N_1})^K_{l_\infty(1/\rho)}$, where $(\cdot, \cdot)^K_{l_\infty(1/\rho)}$ is the functor of the real interpolation method, generated by the parameter $l_\infty(1/\rho)$ (see, e.g., [O] Section 7.1, p. 421). Considering the spaces L_1 and L_{N_1} as Marcinkiewicz spaces and applying [O], Ex. 7.1.3, p. 428 (compare with the proof of Theorem 4.5), we obtain $(L_1, L_{N_1})^K_{l_\infty(1/\rho)} = M_{\tilde{\psi}}$. ∎

6. Comparing independent sums to disjoint sums in the general case

Here we consider the main question (see Section 1) in the setting when the condition (1.1) is no longer assumed. We shall show that the condition $X \in \mathbb{K}$ remains sufficient for a (modified) inequality between the sums of independent random variables and their disjoint copies to hold. For an arbitrary r.i. space X on $[0,1]$ and an arbitrary $p \in [1,\infty]$, we define a function space Z_X^p on $[0,\infty)$ by

$$Z_X^p := \{f \in L_1[0,\infty) + L_\infty[0,\infty) \colon \|f\|'_{Z_X^p} := \|f^*\chi_{[0,1]}\|_X + \|f^*\chi_{[1,\infty)}\|_p < \infty\}.$$

Clearly, $\|\cdot\|'_{Z_X^p}$ is a quasi-norm. It is easy to see that Z_X^p equipped with the equivalent norm

$$\|f\|_{Z_X^p} := \|f^*\chi_{[0,1]}\|_X + \|f\|_{(L_1+L_p)(0,\infty)}, \quad f \in Z_X^p$$

is an r.i. space on $[0,\infty)$. Indeed, the equivalence of the quasi-norm $\|\cdot\|'_{Z_X^p}$ and the norm $\|\cdot\|_{Z_X}$ follows from the well-known formula

$$\|f\|_{(L_1+L_p)(0,\infty)} \asymp \int_0^1 f^*(x)dx + (\int_1^\infty (f^*(x))^p dx)^{1/p}, \quad f \in (L_1+L_p)(0,\infty),$$

where the second summand vanishes when $p=\infty$ and the equivalence constants do not depend on $p \in [1,\infty)$ (see [BL]).

The space $(Z_X, \|\cdot\|_{Z_X}) := (Z_X^1, \|\cdot\|_{Z_X^1})$ was introduced in [JS] and our first result in this section complements [JS] Theorem 1 (see inequality (4) there).

THEOREM 6.1: *Let X and Y be r.i. spaces on $[0,1]$ such that $X \subseteq Y$. If either (i) the operator $\mathcal{K}$ acts boundedly from X into $Y^{\times\times}$ and Y has Fatou norm, or (ii) the operator $\mathcal{K}$ acts boundedly from X into Y, then there exists $C_1 > 0$ such that for every sequence $\{g_i\}_{i=1}^n \subset X$, $n \in \mathbb{N}$, of independent random variables, the following inequality holds:*

$$\left\|\sum_{i=1}^n g_i\right\|_Y \le C_1 \left\|\sum_{i=1}^n \bar{g}_i\right\|_{Z_X}. \tag{6.1}$$

Proof: Without loss of generality we may (and shall) assume that $g_i \geq 0$ and that $\lambda\{g_i = \tau\} = 0$ for all $\tau \in \mathbb{R}$ and all $i = 1, 2, \ldots, n$. Fix $0 = t_n < t_{n-1} < \cdots < t_1 < t_0 = \infty$ such that

$$\sum_{i=1}^{n} \lambda\{t_j < g_i < t_{j-1}\} = 1, \quad j = 1, 2, \ldots, n.$$

For the sequence $\{g_i \chi_{\{g_i > t_1\}}\}_{i=1}^n$ condition (1.1) is satisfied, hence by Theorem 3.5

$$\begin{aligned} \left\| \sum_{i=1}^{n} g_i \chi_{\{g_i > t_1\}} \right\|_Y &\leq C \left\| \sum_{i=1}^{n} \bar{g}_i \chi_{\{\bar{g}_i > t_1\}} \right\|_X \\ &= C \left\| \left(\sum_{i=1}^{n} \bar{g}_i \right)^* \chi_{[0,1]} \right\|_X \leq C \left\| \sum_{i=1}^{n} \bar{g}_i \right\|_{Z_X}. \end{aligned} \tag{6.2}$$

Similarly, condition (1.1) is also satisfied for sequences $\{g_i \chi_{\{t_j < g_i < t_{j-1}\}}\}_{i=1}^n$ $(j = 2, 3, \ldots n)$, hence

$$\begin{aligned} \left\| \sum_{i=1}^{n} g_i \chi_{\{g_i < t_1\}} \right\|_Y \leq & \sum_{j=2}^{n} \left\| \sum_{i=1}^{n} g_i \chi_{\{t_j < g_i < t_{j-1}\}} \right\|_Y \\ \leq & C \sum_{j=2}^{n} \left\| \sum_{i=1}^{n} \bar{g}_i \chi_{\{t_j < \bar{g}_i < t_{j-1}\}} \right\|_X \\ \leq & C \|\chi_{[0,1]}\|_X \sum_{j=2}^{n} t_{j-1} \\ \leq & C \Bigg(\left\| (\sum_{i=1}^{n} \bar{g}_i)^* \chi_{[0,1]} \right\|_X \\ & + \|\chi_{[0,1]}\|_X \sum_{j=2}^{n-1} \left\| \left(\sum_{i=1}^{n} \bar{g}_i \right)^* \chi_{\{t_j < \bar{g}_i < t_{j-1}\}} \right\|_1 \Bigg) \\ = & C \Bigg(\left\| \left(\sum_{i=1}^{n} \bar{g}_i \right)^* \chi_{[0,1]} \right\|_X \\ & + \|\chi_{[0,1]}\|_X \sum_{j=2}^{n-1} \left\| \left(\sum_{i=1}^{n} \bar{g}_i \right)^* \chi_{[j-1,j]} \right\|_1 \Bigg) \\ \leq & C \max\{1, \|\chi_{[0,1]}\|_X\} \left\| \sum_{i=1}^{n} \bar{g}_i \right\|_{Z_X}. \end{aligned}$$

Combining this estimate with (6.2) we arrive at (6.1). ∎

We can now complement Theorem 3.5, Corollary 4.2, Corollary 4.6 and Theorem 5.1 as follows.

COROLLARY 6.2: *The following condition is equivalent to conditions* (i)–(iii) *in Theorem 3.5:*

(iv) *the inequality (6.1) holds for every sequence* $\{g_i\}_{i=1}^n \subset X$, $n \in \mathbb{N}$, *of independent random variables.*

COROLLARY 6.3: *If* Φ *is an Orlicz function such that for some constant* $B \geq 1$,

$$\Phi(x+y) \leq B\Phi(x)\Phi(y), \quad \forall x, y > 0,$$

then there exists $C > 0$ *such that for any finite sequence of independent random variables* $\{f_k\}_{k=1}^n \subset L_\Phi$ *we have*

$$\left\| \sum_{k=1}^n f_k \right\|_{L_\Phi} \leq C \left\| \sum_{k=1}^n \bar{f}_k \right\|_{Z_{L_\Phi}}.$$

COROLLARY 6.4: *For every* $p \in (1, \infty]$ *there exists a constant* $C(p)$ *such that for every sequence* $\{g_i\}_{i=1}^n \subset \exp(L_p)$, $n \in \mathbb{N}$, *of independent random variables, we have*

$$\left\| \sum_{i=1}^n g_i \right\|_{L^0_{M_q}} \leq C \left\| \sum_{i=1}^n \bar{g}_i \right\|_{Z_{\exp(L_p)}}, \quad 1/q + 1/p = 1.$$

COROLLARY 6.5: *Let* $\psi \in \Psi$. *There exists* $C' > 0$ *such that, for any finite sequence of independent random variables* $\{f_k\}_{k=1}^n \subset \Lambda_\psi$, *we have*

$$\left\| \sum_{k=1}^n f_k \right\|_{\Lambda_\psi} \leq C' \left\| \sum_{k=1}^n \bar{f}_k \right\|_{Z_{\Lambda_\psi}}$$

if and only if there exists $C > 0$ *such that (5.1) holds.*

The next corollary extends [JS], inequality (10) which was proved there under the assumption that the r.i. space X contains an L_p-space, $p < \infty$.

COROLLARY 6.6: *Let* X *be an interpolation space for the Banach couple* $(L_1(0,1), L_\infty(0,1))$. *If the operator* $\mathcal{K}$ *acts boundedly on* X, *then there exists a constant* $C > 0$ *such that for every sequence* $\{g_k\}_{k=1}^n \subset X$, $g_k \geq 0$, $1 \leq k \leq n$, $n \in \mathbb{N}$ *and every sequence* $\{f_k\}_{k=1}^n$ *of independent random variables such that* $g_k^* = f_k^*$, $1 \leq k \leq n$, *the following inequality holds:*

$$\left\| \sum_{i=1}^n f_i \right\|_X \leq C \left\| \sum_{i=1}^n g_i \right\|_X.$$

Proof: It follows from [CDS], Lemma 2.3 that for any sequence $\{h_i\}_{i=1}^n$ of non-negative functions from $S(0,1)$, we have

$$\int_0^t \left(\sum_{k=1}^n \bar{h}_k\right)^*(s)ds \le \int_0^t \left(\sum_{k=1}^n h_k\right)^*(s)ds, \quad \forall t>0.$$

It is easy to infer from the well-known description of interpolation spaces for a couple (L_1, L_∞) (see, e.g., [KPS], Theorem $II.4.3$) that the space Z_X is an interpolation space for the Banach couple $(L_1(0,\infty), L_\infty(0,\infty))$. Without loss of generality, we may (and shall) assume that the interpolation constant of Z_X is equal to 1. Therefore, it follows from the inequality above, our assumptions on f_k's and g_k's and [KPS], Theorem $II.4.3$ that

$$\left\|\sum_{k=1}^n \bar{f}_k\right\|_{Z_X} \le \left\|\sum_{k=1}^n g_k\right\|_X.$$

A combination of this inequality with Theorem 6.1 completes the proof. ∎

In the remainder of this section, we shall show how our methods may be used to complement results from [M].

Following [LT] p. 46, we define the space $\widetilde{X(l_p)}$ as the set of all sequences $\mathbf{f} = \{f_k(x)\}_{k=1}^\infty$, $f_k \in X$, $k \ge 1$ such that

$$\|\mathbf{f}\|_{\widetilde{X(l_p)}} := \sup_n \left\|\left(\sum_{k=1}^n |f_k|^p\right)^{1/p}\right\|_X < \infty$$

(with an obvious modification for $p=\infty$). The closed subspace of $\widetilde{X(l_p)}$ generated by all eventually vanishing sequences $\mathbf{f} \in \widetilde{X(l_p)}$ is denoted by $X(l_p)$.

Before proceeding, we recall the following construction due to A. P. Calderon [C]. Let X_0 and X_1 be two Banach lattices of measurable functions on the same measure space $(\mathcal{M}, m)$ and let $\theta \in (0,1)$. The space $X_0^{1-\theta}X_1^\theta$ consists of all measurable functions f on $(\mathcal{M}, m)$ such that for some $\lambda > 0$ and $f_i \in X_i$ with $\|f_i\|_{X_i} \le 1$, $i = 0,1$,

$$|f(x)| \le \lambda |f_0(x)|^{1-\theta}|f_1(x)|^\theta, \quad x \in \mathcal{M}$$

equipped with the norm given by the greatest lower bound of all numbers λ taken over all possible such representations. Even though this construction is not an interpolation functor on general couples of Banach lattices (see [Lo]), it is still a convenient tool of interpolation theory. Indeed, if (X_0, X_1) is a Banach couple and if (Y_0, Y_1) is another Banach couple of lattices of measurable functions on some measure space $(\mathcal{M}', m')$, then any positive operator A from $S(\mathcal{M}, m)$ into $S(\mathcal{M}', m')$, which acts boundedly from the couple (X_0, X_1) into the couple (Y_0, Y_1), also maps boundedly $X_0^{1-\theta} X_1^{\theta}$ into $Y_0^{1-\theta} Y_1^{\theta}$ and, in addition, $\|A\|_{X_0^{1-\theta} X_1^{\theta} \to Y_0^{1-\theta} Y_1^{\theta}} \leq \|A\|_{X_0 \to Y_0}^{1-\theta} \|A\|_{X_1 \to Y_1}^{\theta}$ for all $\theta \in (0,1)$. The proof of the latter claim follows by inspection of the standard arguments from [LT], Proposition 1.d.2(i), p. 43.

The following theorem is proved in [M], in the special case that $X = Y$ and $L_q \subseteq X$ for some $q < \infty$. At the same time the result of [M] is concerned with sequences of random variables in a general symmetric sequence space, whereas we consider here the case of l_p-spaces only.

THEOREM 6.7: *Let X and Y be r.i. spaces on $[0,1]$ such that $X \subseteq Y$ and let $p \in [1, \infty]$. If either (i) the operator $\mathcal{K}$ acts boundedly from X into $Y^{\times\times}$ and Y has Fatou norm, or (ii) the operator $\mathcal{K}$ acts boundedly from X into Y, then there exists $C > 0$ which depends on X and Y only such that for every sequence $\mathbf{g} = \{g_i\}_{i=1}^n \subset X$, $n \in \mathbb{N}$, of independent functions, the following inequality holds:*

$$(6.1)' \qquad \|\mathbf{g}\|_{Y(l_p)} \leq C \left\| \sum_{i=1}^n \bar{g}_i \right\|_{Z_X^p}.$$

Proof: Let T be the rearrangement-preserving mapping between $S(\Omega, \mathcal{P})$ and $S([0,1], \lambda)$ introduced in Section 3. We define the positive linear mapping Q from $S(0, \infty)$ into $S(\Omega, \mathcal{P})^{\mathbb{N} \cup \{0\}}$ by setting

$$(Qf(\omega_0, \omega_1, \ldots) := \{f_k(\omega_k)\}_{k=0}^{\infty}, \quad f \in S(0, \infty),$$

where $f_k(\omega_k) := f(\omega_k + k)$, $k \geq 0$. The proof of Theorem 6.7 will be completed as soon as we show that the positive linear operator $Q' := TQ$ is a bounded linear operator from Z_X^p into $Y(l_p)$. The key observations are that the operator Q' acts boundedly from $Z_X^1 = Z_X$ into $Y(l_1)$ and from Z_X^{∞} into $Y(l_{\infty})$. Indeed, the first observation follows immediately from Theorem 6.1 (if one takes into account that for every sequence $\mathbf{g} = \{g_i\}_{i=1}^n \subset X$, $n \in \mathbb{N}$, of independent functions in X, the sequence $|\mathbf{g}| := \{|g_i|\}_{i=1}^n \subset X$ is again a sequence of independent functions and that $\| \sum_{i=1}^n |\bar{g}_i| \|_{Z_X} = \| \sum_{i=1}^n \bar{g}_i \|_{Z_X}$). The second observation follows from a combination of the equivalences

$$\|f\|_{Z_X^{\infty}} \asymp \|f\|'_{Z_X^{\infty}} \asymp \|f^* \chi_{[0,1]}\|_X, \quad \forall f \in Z_X^{\infty}$$

(where the equivalence constants do not depend on the r.i. space X and $f \in Z_X^{\infty}$)

with the inequalities

$$\frac{1}{2}\lambda\{\bar{f}^*\chi_{[0,1]} > \tau\} \le \lambda\{\max_{k=1,2,\dots,n}|f_k| > \tau\} \le \lambda\{\bar{f}^*\chi_{[0,1]} > \tau\}, \quad \forall\tau > 0,$$

where $\{f_k\}_{k=1}^n \subset X$, $n \in \mathbb{N}$, is a sequence of independent random variables in X and $\bar{f} := \sum_{k=1}^n \bar{f}_k$ (see [HM], Proposition 2.1).

It follows that

$$Q'\colon\ (Z_X^1)^{1-\theta}(Z_X^\infty)^\theta \to (Y(l_1))^{1-\theta}(Y(l_\infty))^\theta$$

and its norm is uniformly bounded with respect to $\theta \in (0,1)$. The proof is complete by noting that for all $\theta \in (0,1)$ we have

$$Z_X^p \subseteq (Z_X^1)^{1-\theta}(Z_X^\infty)^\theta, \quad (Y(l_1))^{1-\theta}(Y(l_\infty))^\theta \subseteq Y(l_p), \quad p = \frac{1}{1-\theta}. \tag{6.3}$$

To see the first embedding above, fix $g = g^* \in Z_X^p$, $\|g\|_{Z_X^p} = 1$ and set

$$g_1 := g\chi_{[0,1]} + g^p\chi_{[1,\infty)}, \quad g_\infty := g\chi_{[0,1]} + \chi_{[1,\infty)}.$$

Clearly, $g = (g_1)^{1-\theta}(g_\infty)^\theta$ and it is a straightforward verification that $g_i \in Z_X^i$ and $\|g_i\|_{Z_X^i} \le C$, $i = 1,\infty$, where $C > 0$ does not depend on p. The second embedding in (6.3) (in fact, equality) is shown in [Bu], Theorem 3. ∎

Remark 6.8: The assertion established in Theorem 6.7 follows from the boundedness of a certain linear operator from Z_X^p into $Y(l_p)$, which is a consequence of the boundedness of this operator from the couple (Z_X^1, Z_X^∞) into the couple $(Y(l_1), Y(l_\infty))$. By using Calderon–Lozanovskii's construction (see, e.g., [O], Section 8.2 and also [Bu]), it is possible to extend this result to more general spaces than Z_X^p and $Y(l_p)$, but we have not pursued this subject in the present paper.

7. Final remarks

It follows from Corollaries 5.6 and 5.7 that the assumption $\exp(L_1) \subseteq X$ is not sufficient for an r.i. space X to have the Kruglov property. We shall present a concrete example of a Lorentz space Λ_ψ without the Kruglov property satisfying $\exp(L_1) \subseteq \Lambda_\psi$.

PROPOSITION 7.1: *If $\kappa_\beta(t) := \ln^{-1}\frac{e}{t}\cdot\ln^{-\beta}(\ln\frac{e^e}{t})$, $t \in [0,1]$, $\beta > 1$, then $\exp(L_1) \subseteq \Lambda_{\kappa_\beta}$, but the latter space does not have the Kruglov property.*

Proof: It is sufficient to verify that the function κ_β does not satisfy condition (5.1) for every $\beta > 1$. Such a verification is straightforward but technical, and we omit detailed calculations. ∎

We present now some necessary condition for a r.i. space X to have the Kruglov property.

For $n \in \mathbb{N}$, we denote the nth repeated logarithm by $\ln_n$ and set

$$\Phi_n(u) := \exp(u \ln_n(c_n + u)) - 1, \quad \ln_n c_n = 1.$$

The Orlicz space L_{Φ_n} coincides with the Marcinkiewicz space M_{ξ_n}, where

$$\xi_n(u) := \frac{u \ln(e/u)}{\ln_{n+1}(e^{c_n}/u)}, \quad u \in (0,1].$$

Let X be an r.i. space. The assumption $\exp(L_1) \subseteq X$ seems to be necessary for the operator K to act boundedly on X. The following result is a step towards proving this conjecture.

THEOREM 7.2: *If the operator $\mathcal{K}$ acts boundedly on the r.i. space X, then $L_{\Phi_n} \subset X$ for every $n \in \mathbb{N}$.*

Proof: Since $L_\infty \subseteq X$, it follows from Theorem 4.4 that

$$f_1(\cdot) := \frac{\ln(e/\cdot)}{\ln(\ln(e^e/\cdot))} \in X.$$

Noting that $\lambda\{f_1 > \tau\} \asymp e^{-\tau \ln(1+\tau)}$ and arguing as in the proof of Theorem 4.5, we get

$$\lambda\{\mathcal{K} f_1 > \tau\} \geq C \sum_{n=1}^{\infty} e^{-\tau \ln(1+\tau/n)} \frac{1}{n!}.$$

If $n > \tau \ln^{-1}(\tau/e)$, then $\tau \ln(1+\tau/n) < \tau \ln(\ln(\tau))$ and therefore

$$\lambda\{\mathcal{K} f_1 > \tau\} \geq C e^{-\tau \ln(\ln(\tau))} \sum_{n \geq [2\tau \ln^{-1}(\tau/e)]} 1/n!. \tag{7.1}$$

Using Stirling's formula for sufficiently large τ's we estimate

$$\frac{1}{[2\tau \ln^{-1} \frac{\tau}{e}]!} \geq \exp(-2\tau \ln^{-1} \frac{\tau}{e} \ln(2\tau \ln^{-1} \frac{\tau}{e})) \geq e^{-3\tau}.$$

Applying (7.1), we obtain for all sufficiently large τ

$$\lambda\{\mathcal{K} f_1 > \tau\} \geq C e^{-\tau \ln(\ln(\tau))} e^{-3\tau} \geq e^{-4\tau \ln(\ln(\tau))}. \tag{7.2}$$

Note that for the Marcinkiewicz space M_{ξ_n}, $n \geq 1$ the analogue of formula (4.1) holds. Therefore, to prove that M_{ξ_2} is contained in X, it is sufficient to show that the function $f_2(t) := t \to \xi_2(t)/t$ belongs to X. In particular, it is sufficient to verify that there exists a constant $C > 0$ such that for all sufficiently large τ,

$$\lambda\{f_2 > \tau\} \leq C\lambda\{\mathcal{K}f_1 > \tau\}. \tag{7.3}$$

Direct calculations now show that for sufficiently large τ we have $\lambda\{f_2 > \tau\} \leq Ce^{-4\tau\ln(\ln(\tau))}$, i.e. (7.3) now follows from (7.2). Repeating the same arguments and calculations for the function $\mathcal{K}f_2$ we infer further that the function $f_3(t) := t \to \xi_3(t)/t$ belongs to X and so on. This completes the proof of Theorem 7.2. ∎

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